Doubly Stochastic Matrices Which Have Certain Diagonals with Constant Sums*

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ABSTRACT

Let A be an $n\times n$ doubly stochastic matrix and suppose that $1\leqslant m\leqslant n-1$. Let $\tau_1,\dots,\tau_m$ be m mutually disjoint zero diagonals in A, and suppose that every diagonal of A disjoint from $\tau_1,\dots,\tau_m$ has a constant sum. Then all entries of A off the m zero diagonals have the value $(n-m)^{-1}$. This verifies a conjecture of E. T. Wang.

Let S_n denote the full symmetric group of degree n. Let A be a real $n\times n$ matrix. For $\sigma\in S_n$, the set $\{a_{1\sigma(1)}, a_{2\sigma(2)},\dots,a_{n\sigma(n)}\}$ is called the *diagonal* of A corresponding to σ. For convenience the diagonal itself is denoted by σ.

Two diagonals σ,τ in A are said to be *disjoint* if $\sigma(i)\neq\tau(i)$ for $i=1,\dots,n$.

If E and F are each nonvoid subsets of $N=\{1,\dots,n\}$, then $A[E|F]$ denotes the submatrix of A whose rows are indexed by E and whose columns are indexed by F each in the same order as they appear in A. Also, $A(E|F]$, $A[E|F)$, and $A(E|F)$ are used to denote $A[N\backslash E|F]$, $A[E|N\backslash F]$, and $A[N\backslash E|N\backslash F]$, respectively.

The $n\times n$ matrix $A=(a_{ij})$ is said to be *doubly stochastic* if every $a_{ij}\geqslant 0$ and if $\sum_{k=1}^{n} a_{ik}=\sum_{k=1}^{n} a_{kj}=1$ for all $i,j=1,\dots,n$. The set of $n\times n$ doubly stochastic matrices is denoted by Ω_n.

In [2], Wang makes the conjecture stated below in Theorem 1. Wang proves Theorem 1 for $m=1$, $n-2$, and $n-1$. We prove it for all $m=1,\dots,n-1$.

THEOREM 1. *Let $A\in\Omega_n$, and let $\tau_1, \tau_2,\dots,\tau_m$ be m mutually disjoint zero diagonals of A, $1\leqslant m\leqslant n-1$. If every diagonal disjoint from each τ_i,*

*This paper is dedicated to my beloved wife, Mona Jean Sinkhorn.

$i=1,\ldots,m$, has a constant sum, then all entries off the m zero diagonals are equal to $(n-m)^{-1}$.

An important tool needed to prove the result is the following. It is an immediate consequence of Corollary 2 in [1].

THEOREM 2. *If A is a nonnegative $n\times n$ matrix with a zero pattern identical to that of some $B\in\Omega_n$ (i.e., $a_{ij}=0\Leftrightarrow b_{ij}=0$), and if the positive diagonal products of A are all equal, then there exist positive numbers $r_1,\ldots,r_n$ and $c_1,\ldots,c_n$ such that whenever $a_{ij}>0$, $a_{ij}=r_ic_j$.*

We now prove Theorem 1.

Define $B=(b_{ij})$ as follows. Put $b_{ij}=0$ if a_{ij} is on any of the diagonals $\tau_1,\ldots,\tau_m$, and $b_{ij}=e^{a_{ij}}$ otherwise. B has a zero pattern identical to that of the doubly stochastic matrix in the conclusion to Theorem 1. The matrix B has its positive diagonal products equal, and thus by Theorem 2, there exist positive numbers $r_1,\ldots,r_n$, $c_1,\ldots,c_n$ such that $b_{ij}=r_ic_j$ whenever $b_{ij}>0$. Hence $a_{ij}=\ln r_i+\ln c_j=\alpha_i+\beta_j$ off $\tau_1,\ldots,\tau_m$.

Put $E_j=\{i|a_{ij}\notin\tau_k,\ k=1,\ldots,m\}$, $F_i=\{j|a_{ij}\notin\tau_k,\ k=1,\ldots,m\}$. Set $\bar{\alpha}=\min\alpha_i$, $\bar{\beta}=\max\beta_j$, and let $E=\{i|\alpha_i=\bar{\alpha}\}$, $F=\{j|\beta_j=\bar{\beta}\}$. Observe in particular that each E_j and F_i has cardinality $n-m$.

Since $A\in\Omega_n$,

$$(n-m)\alpha_i+\sum_{k\in F_i}\beta_k=1,\qquad i=1,\ldots,n,\tag{1}$$

and

$$(n-m)\beta_j+\sum_{i\in E_j}\alpha_i=1,\qquad j=1,\ldots,n.\tag{2}$$

Thus

$$(n-m)\sum_{i\in E_j}\alpha_i+\sum_{i\in E_j}\sum_{k\in F_i}\beta_k=n-m,$$

and so

$$(n-m)\left[1-(n-m)\beta_j\right]+\sum_{i\in E_j}\sum_{k\in F_i}\beta_k=n-m,\qquad j=1,\ldots,n,$$

or

$$\beta_j = (n-m)^{-2} \sum_{i \in E_j} \sum_{k \in F_i} \beta_k, \qquad j = 1, \ldots, n. \tag{3}$$

If $j_0 \in F$,

$$\bar{\beta} = \beta_{j_0} = (n-m)^{-2} \sum_{i \in E_{j_0}} \sum_{k \in F_i} \beta_k \leqslant (n-m)^{-2}(n-m)^2 \bar{\beta} = \bar{\beta}. \tag{4}$$

It follows that $\beta_k = \bar{\beta}$ for all $k \in F_i$ if $i \in E_j$ and $j \in F$. From (1), if $i \in E_j$, where $j \in F$, then $\alpha_i = (n-m)^{-1} - \bar{\beta}$. However, it is also clear from (1) that for any i, $\alpha_i \geqslant (n-m)^{-1} - \bar{\beta}$. Whence $\bar{\alpha} = (n-m)^{-1} - \bar{\beta}$, i.e., $\bar{\alpha} + \bar{\beta} = (n-m)^{-1}$.

Again by (1) it is seen that if $i \in E$, and thus if $\alpha_i = \bar{\alpha} = (n-m)^{-1} - \bar{\beta}$, then

$$1 - (n-m)\bar{\beta} + \sum_{k \in F_i} \beta_k = 1,$$

and so all $\beta_k = \bar{\beta}$ when $k \in F_i$. This means that

$$\bigcup_{i \in E} F_i \subseteq F. \tag{5}$$

By a symmetrical argument using (2) one derives

$$\bigcup_{j \in F} E_j \subseteq E. \tag{6}$$

If $E \subset N$ properly, it follows from (6) that a_{ij} belongs to some τ_k if $i \notin E$ and $j \in F$, and therefore that $A(E|F] = 0$. Similarly, if $F \subset N$ properly, it follows from (5) that $A[E|F) = 0$ with every entry on some τ_k. In either case, since $A \in \Omega_n$, it is seen that $A[E|F]$ is doubly stochastic and is thus a square matrix. For $(i,j) \in E \times F$, $a_{ij} = \bar{\alpha} + \bar{\beta} = (n-m)^{-1}$ off $\tau_1, \ldots, \tau_m$. Hence if $E = F = N$, the proof is complete. If this is not the case we simply repeat the argument on the doubly stochastic matrix $A(E|F)$. Since $A(E|F]$ and $A[E|F)$ are completely covered by $\tau_1, \ldots, \tau_m$, the sets E_j and F_i are left unchanged when (i,j) is restricted to $N \backslash E \times N \backslash F$. The numbers $\bar{\alpha}$ and $\bar{\beta}$ are replaced by the second smallest α_i and the second largest β_j, respectively, and the sets E and F are changed accordingly. The results in (1)–(6) will appear exactly as they did before. If necessary the process can be repeated again. After a finite number of steps one obtains the desired result.

It is of interest to consider an example:

$$A=\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix},$$

in which $n=6$, $m=4$, and $\tau_1:(1\ 2\ 3\ 4\ 5\ 6)\to(3\ 4\ 5\ 6\ 1\ 2)$, $\tau_2:(1\ 2\ 3\ 4\ 5\ 6)\to(4\ 5\ 6\ 3\ 2\ 1)$, $\tau_3:(1\ 2\ 3\ 4\ 5\ 6)\to(5\ 6\ 1\ 2\ 3\ 4)$, $\tau_4:(1\ 2\ 3\ 4\ 5\ 6)\to(6\ 1\ 2\ 5\ 4\ 3)$. It turns out that $E=F=\{1,2,3,4\}$ if one uses $\alpha_1=\alpha_2=\alpha_3=\alpha_4=0$, $\beta_1=\beta_2=\beta_3=\beta_4=\frac{1}{2}$, and $\alpha_5=\alpha_6=\beta_5=\beta_6=\frac{1}{4}$. In this case $A[E|F]$ cannot be decomposed into a direct sum of more than one doubly stochastic matrix. The τ_k induce a set of zero entries in $A[E|F]$ which can be completely covered by two disjoint diagonals $\sigma_1:(1\ 2\ 3\ 4)\to(3\ 4\ 1\ 2)$ and $\sigma_2:(1\ 2\ 3\ 4)\to(4\ 1\ 2\ 3)$, each of which came as a result of *two* of the τ_k. The τ_k induce no zero entries in $A(E|F)$.

If, however, one had picked $\alpha_i=\beta_i=\frac{1}{4}$, $i=1,\dots,6$, for example, then $E=F=N=\{1,2,3,4,5,6\}$. In this case $A[E|F]=A$ certainly can be decomposed into a direct sum of more than one doubly stochastic matrix.

REFERENCES

1 Richard Sinkhorn and Paul Knopp, Problems involving diagonal products in nonnegative matrices, *Trans. Am. Math. Soc.* **136** (1969), 67–75.

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